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Smirnov Domains and Conjugate Functions*

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Let $f(z)$ map the unit disk $|z| < 1$ conformally onto a domain D bounded by a rectifiable Jordan curve C. Then f' belongs to the Hardy class H^1, so it has a canonical factorization of the form

$$f'(z) = e^{i\gamma}S(z)\,G(z). \tag{1}$$

Here γ is a real number;

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\sigma(t)\right\},$$

where σ is a bounded nondecreasing singular function: $\sigma'(t) = 0$ a.e.; and

$$G(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log|f'(e^{it})|\,dt\right\}.$$

D is called a *Smirnov domain* if $S(z) \equiv 1$; that is, if $d\sigma$ is the zero measure. This is a property only of D, not of f [1, Chapter 10].

Smirnov domains are known to play an important role in the theory of polynomial approximation and orthogonal expansion in the complex plane. If D is any Jordan domain, Walsh's theorem tells us that each function analytic in D and continuous in $\bar{D}$ can be approximated uniformly in $\bar{D}$ by a polynomial. However, the L^p analogue $(0 < p < \infty)$ of this theorem is true if and only if D is a Smirnov domain.

This last statement has several interpretations. To be more specific, we shall introduce some notation. Let D be the interior of a rectifiable

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Jordan curve C, and let $L^p(C)$ be the class of complex-valued functions f for which $|f(z)|^p$ is integrable over C with respect to arclength. Let $L^\infty(C)$ be the class of bounded measurable functions on C. For $1 \leqslant p \leqslant \infty$, let $A^p(C)$ be the class of all $f \in L^p(C)$ whose Cauchy integral

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

vanishes identically outside C. For $0 < p < \infty$, let $E^p(D)$ be the class of functions f analytic in D, for which there is a sequence $\{C_n\}$ of rectifiable Jordan curves in D, tending to C in the sense that C_n eventually surrounds each compact subset of D, such that

$$\sup_n \int_{C_n} |f(z)|^p \, |dz| < \infty.$$

Let $E^\infty(D)$ be the class of bounded analytic functions in D. Each $f \in E^p(D)$ has a nontangential limit almost everywhere on C, and the boundary function belongs to $L^p(C)$. Let $E^p(C)$ be the class of all such boundary functions. Finally, let $\pi^p(C)$ be the closure in $L^p(C)$ of the polynomials.

For any rectifiable Jordan curve C, it is clear that $\pi^p(C) \subset E^p(C)$, $0 < p < \infty$. The question of equality is answered by the following theorem.

THEOREM A. *Let C be a rectifiable Jordan curve, and let D be its interior. Then for each p, $0 < p < \infty$, $\pi^p(C) = E^p(C)$ if and only if D is a Smirnov domain.*

This result is essentially due to Smirnov [10], who considered only the case $p = 2$. Keldysh [5] apparently was the first to state it for general p. A proof using Beurling's approximation theorem may be found in [1].

For any rectifiable Jordan curve C, it can be proved that $E^1(C) = A^1(C)$. This result also goes back to Smirnov [1, Theorem 10.4]. Since $E^p(C) \subset E^1(C)$ for all $p > 1$, it follows at once that $E^p(C) \subset A^p(C)$, $1 < p \leqslant \infty$. It seems remarkable that for $p > 1$, $A^p(C)$ can actually be larger than $E^p(C)$. In fact, we have the following theorem.

THEOREM 1. *Let C be a rectifiable Jordan curve, and let D be its interior. Then for each p, $1 < p \leqslant \infty$, $E^p(C) = A^p(C)$ if and only if D is a Smirnov domain.*

The proof is based on another theorem which is of independent interest. For H^p spaces in the unit disk, it is familiar that if $f \in H^p$ and its boundary function belongs to L^q for some $q > p$, then $f \in H^q$ [1, Theorem 2.11]. This statement can be generalized as follows.

THEOREM 2. *Let C be a rectifiable Jordan curve, and let D be its interior. Then for each pair (p, q) with $0 < p < q \leqslant \infty$, $E^p(C) \cap L^q(C) = E^q(C)$ if and only if D is a Smirnov domain.*

Proof of Theorem 2. Let $z = \varphi(w)$ map the unit disk $|w| < 1$ conformally onto D, and suppose $\varphi'(0) > 0$. Let $w = \psi(z)$ be the inverse mapping. Then $f \in E^p(D)$ if and only if

$$F(w) = f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p$$

[1, p. 169]. If $f \in L^q(C)$, then

$$F_1(w) = F(w)[\varphi'(w)]^{1/q-1/p} = f(\varphi(w))[\varphi'(w)]^{1/q}$$

has a boundary function of class L^q. But if D is a Smirnov domain, then φ' has no singular factor, and $F_1 \in N^+$ [1, p. 26]. Thus [1, Theorem 2.11] $F_1 \in H^q$, which proves $f \in E^q(D)$.

Conversely, suppose D is not a Smirnov domain. Let $\varphi' = SG$ be the canonical factorization of the form (1), and consider the function

$$g(z) = [S(\psi(z))]^{-1} \tag{2}$$

It is clear that $g \in L^\infty(C)$. We claim that $g \in E^1(D)$, but $g \notin E^p(D)$ for all $p > 1$. Indeed,

$$g(\varphi(w))\, \varphi'(w) = G(w) \in H^1,$$

but

$$g(\varphi(w))[\varphi'(w)]^{1/p} = [S(w)]^{1/p-1}[G(w)]^{1/p} \notin H^p$$

if $p > 1$. Thus for given p and q, $0 < p < q \leqslant \infty$, $[g]^{1/p} \in E^p(C) \cap L^q(C)$, but $[g]^{1/p} \notin E^q(C)$. This proves Theorem 2.

Proof of Theorem 1. If $p > 1$, then $A^p(C) \subset A^1(C) = E^1(C)$ and $A^p(C) \subset L^p(C)$. If D is a Smirnov domain, Theorem 2 allows us to conclude that $A^p(C) \subset E^p(C)$. But since the reverse inclusion holds for every rectifiable Jordan curve C, this implies $A^p(C) = E^p(C)$.

Now suppose that D is not a Smirnov domain, and again consider the function g defined in (2). We have already seen that $g \in L^\infty(C)$ but $g \notin E^p(C)$ if $p > 1$. On the other hand,

$$\int_C g(z)\, z^n\, dz = \int_{|w|=1} G(w)[\varphi(w)]^n\, dw = 0, \qquad n = 0, 1, \ldots,$$

since $G \in H^1$ and $\varphi \in H^\infty$. This shows $g \in A^p(C)$ for all p, $1 \leqslant p \leqslant \infty$. Hence the proof of Theorem 1 is complete.

These results indicate the importance of finding useful conditions for a Jordan domain D with rectifiable boundary C to be a Smirnov domain. One sufficient condition is that

$$\log f'(z) \in H^1, \tag{3}$$

where f is a conformal mapping of the unit disk onto D. In particular, D is a Smirnov domain if the local rotation $\arg\{f'(z)\}$ has a one-sided bound. This will be the case, for instance, if D is a starlike domain, or if C is an analytic curve. Tumarkin [11] and Shapiro [8] have found other sufficient conditions.

The question arises whether the condition (3) actually characterizes the Smirnov domains. By means of the following theorem of Duren, Shapiro, and Shields [2], we shall reduce this question to a purely "real-variable" problem.

THEOREM B. *Let $\mu(t)$ be a real-valued left-continuous function of bounded variation over* $[0, 2\pi]$, *and let*

$$\mu(t) = \mu_s(t) + \int_0^t \varphi(\tau)\, d\tau$$

be its canonical decomposition into singular and absolutely continuous components. Let

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t).$$

Then there exists a constant $a > 0$ such that $\exp\{-aF(z)\}$ *is the derivative of a function $f(z)$ which maps the unit disk $|z| < 1$ conformally onto a Jordan domain, if and only if $\mu \in \Lambda_*$. The boundary of this domain is rectifiable if and only if $\mu_s(t)$ is nondecreasing and* $\exp\{-a\varphi(t)\} \in L^1$.

Note. The Zygmund class Λ_* is familiar in approximation theory. A function $\mu(t)$ continuous on $[0, 2\pi]$ is said to belong to Λ_* if its "periodic extension" has the property

$$|\mu(t+h) - 2\mu(t) + \mu(t-h)| \leqslant A\,|h|$$

for some constant A independent of t and h.

In particular, Theorem B shows that the construction of a Jordan domain with rectifiable boundary whose mapping function f has a purely singular derivative (i.e., $f' = S$ in (1)), as in the example of Keldysh and Lavrentiev [6], is equivalent to the construction of a singular nondecreasing bounded function of class Λ_*. Piranian [7], Kahane [3], and Shapiro [9] have carried out this latter construction directly.

Before stating the next theorem, we recall the definition of a conjugate function. If $\varphi \in L^1 = L^1(0, 2\pi)$, then [1, Theorem 4.2] the function

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \varphi(t)\, dt \tag{4}$$

belongs to H^p for all $p < 1$. In particular, $\operatorname{Im}\{F(z)\}$ has a radial limit almost everywhere, which is denoted $\tilde{\varphi}$ and is called the *conjugate function* of φ.

THEOREM 3. *There exists a Smirnov domain D such that* $\log f'(z) \notin H^1$ *for every conformal mapping f of the unit disk onto D.*

This theorem is a consequence of the following lemma, to be proved at the end of the paper.

LEMMA. *There exists a real-valued function* $\varphi(t)$ *on* $0 \leqslant t \leqslant 2\pi$ *such that* $\varphi \in L^1$, $e^{-\varphi} \in L^1$, $\int \varphi \in \Lambda_*$, *and* $\tilde{\varphi} \notin L^1$.

Remark. By $\int \varphi$ is meant the indefinite integral of φ, say,

$$\mu(t) = \int_0^t \varphi(\tau)\, d\tau. \tag{5}$$

The slightly stronger condition $\mu \in \Lambda_1$ would imply $\varphi \in L^\infty$, hence that $\tilde{\varphi} \in L^p$ for all $p < \infty$.

Proof of Theorem 3. Let φ have the properties described in the lemma, let F be the Poisson integral (4), and let μ be the indefinite integral (5). Then since $\mu \in \Lambda_*$, Theorem B says that for some constant $a > 0$,

$$f'(z) = \exp\{-aF(z)\} \tag{6}$$

is the derivative of a conformal mapping f of the unit disk onto a Jordan domain D. Since $\mu_s = 0$ and $e^{-\varphi} \in L^1$, Theorem B also says (if we take $a \leqslant 1$) that D has rectifiable boundary. Finally, since μ is absolutely continuous, it is clear from (6) that D is a Smirnov domain. However, the condition $\tilde{\varphi} \notin L^1$ implies $F \notin H^1$, $\log f'(z) \notin H^1$. But if $\log f'(z) \notin H^1$ for some mapping function f of the disk onto D, then the same is true for every other mapping function. This is easily seen, for example, with the harmonic majorant definition of H^1.

It is interesting to observe that, conversely, given any Smirnov domain with $\log f'(z) \notin H^1$, Theorem B shows that the function

$$\varphi(t) = -\log |f'(e^{it})|$$

has the properties described in the lemma. Thus the lemma is actually equivalent to Theorem 3.

Proof of lemma. The following construction was suggested by Y. Katznelson and K. deLeeuw (private communication). Let $\alpha(t)$ be any bounded singular nondecreasing function of class Λ_*. (Such functions exist, as noted above.) Let

$$f(z) = u(z) + iv(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\alpha(t).$$

Then $u(z) > 0$ and $u \in h^1$, but $f \notin H^1$ since $\alpha(t)$ is not absolutely continuous [1, p. 34]. Thus $\| v_r \|_1 \to \infty$ as $r \to 1$, where $v_r(\theta) = v(re^{i\theta})$ and $\| \ \|_p$ denotes the L^p norm. Let

$$C = \frac{1}{2\pi} [\alpha(2\pi) - \alpha(0)],$$

and define $\beta(t)$ as the periodic extension of

$$\beta(t) = \alpha(t) - Ct, \qquad 0 \leqslant t \leqslant 2\pi.$$

Then an integration by parts gives

$$U_r(\theta) = \int_0^\theta u_r(\theta) \, d\theta = \int_0^{2\pi} P(r, t) \, \beta(\delta + t) \, dt + 2\pi C\theta,$$

where

$$P(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}$$

is the Poisson kernel. This shows that $U_r \in \Lambda_*$ and

$$| U_r(\theta + h) - 2U_r(\theta) + U_r(\theta - h)| \leqslant A \, | h |, \tag{7}$$

where the constant A is independent of r.

Now choose a sequence $\{r_k\}$ increasing to 1, and let

$$\varphi_n(\theta) = \sum_{k=1}^{n} 3^{-k} u_{r_k}(\theta).$$

Then $\varphi_n(\theta) \to \varphi(\theta)$ a.e., $\varphi(\theta) \geqslant 0$, and $\varphi \in L^1$. By the Lebesgue monotone convergence theorem and by (7),

$$\mu(\theta) = \int_0^\theta \varphi(\theta) \, d\theta = \sum_{k=1}^{\infty} 3^{-k} U_{r_k}(\theta) \in \Lambda_* .$$

On the other hand, $f \in H^p$ for all $p < 1$, so

$$\tilde{\varphi}_n(\theta) = \sum_{k=1}^{n} 3^{-k} v_{r_k}(\theta) \to \sum_{k=1}^{\infty} 3^{-k} v_{r_k}(\theta) = \tilde{\varphi}(\theta) \quad \text{a.e.}$$

But $\| v_r \|_1 \to \infty$ as $r \to 1$, and $\| v \|_p \to \infty$ as $p \to 1$. Thus we may choose $\{r_k\}$ and a sequence $\{p_n\}$ of positive numbers increasing to 1, such that $\| v_{r_1} \|_1 > 1$, $3^{p_1} > 5/2$,

$$(\| v \|_{p_n})^{p_n} > 3^{2n} \sum_{k=1}^{n-1} \| v_{r_k} \|_1 > 3^{2n}, \qquad n = 2, 3, \ldots,$$

and

$$(\| v_{r_n} \|_{p_n})^{p_n} > \frac{8}{9} (\| v \|_{p_n})^{p_n}, \qquad n = 1, 2, \ldots .$$

Then for $n > 1$,

$$(\| \tilde{\varphi} \|_{p_n})^{p_n} > 3^{-np_n} (\| v_{r_n} \|_{p_n})^{p_n} - \sum_{k \neq n} 3^{-kp_n} (\| v_{r_k} \|_{p_n})^{p_n}$$

$$> 3^{-n-2} 8 (\| v \|_{p_n})^{p_n} - \sum_{k=1}^{n-1} \| v_{r_k} \|_1 - (\| v \|_{p_n})^{p_n} \sum_{k=n+1}^{\infty} 3^{-kp_n}$$

$$> [3^{-n-2} 8 - 3^{-n-2} - 3^{-np_n} (3^{p_n} - 1)^{-1}] (\| v \|_{p_n})^{p_n}$$

$$> 3^{-n-2} 3^{2n} = 3^{n-2}.$$

Thus $\| \tilde{\varphi} \|_{p_n} \to \infty$ as $n \to \infty$, which shows that $\tilde{\varphi} \notin L^1$. This concludes the proof.

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